with K and T_1, since $(m^2/c)(c/m^2)(-c) = -c$. Therefore to construct K we draw OT_1 perpendicular to PP_1, OK the reflection of OT_1 around the bisector of the angle between the nodal tangents, and T_1K parallel to the medial line g.

It remains to put the pencil of circles on PP_1 into graphical correspondence with the pencil of lines at K. Each circle through P and P_1 cuts the nodal radii OP and OP_1 again in two points R and R_1. To determine R, substitute $y = mx$ in (3). The result is a quadratic in x the product of whose roots is $d/(1 + m^2)$. But one of these is the x in (1); therefore the other root is $d(m + c)/am$. In this way we find the coördinates of R and R_1 to be:

$$\left[\frac{d}{am}(m+c), \quad \frac{d}{a}(m+c)\right] \quad \text{and} \quad \left[\frac{d}{am}(m-c), \quad \frac{-d}{a}(m-c)\right].$$

The equation of the line RR_1 is then

$$am^2x - acy - d(m^2 - c^2) = 0,$$

and it requires only some algebraic drudgery to show that this line meets KQQ_1 on the medial line g.

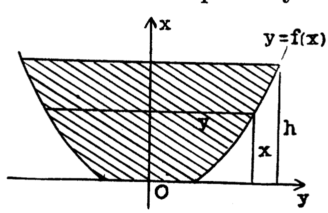
Accordingly, the strophoid may be constructed as follows, given the node and two conjugate points P and P_1. Construct the nodal tangents, the medial line, the line OK and the point K where it will cut the curve. Each circle of the pencil through PP_1 cuts the nodal radii OP, OP_1 in two points R, R_1; the line RR_1 cuts the medial line in a point L, and the line LK cuts the circle in its remaining real intersections with the strophoid.

5. While these constructions are not superior to the classical one in case of actual use on the drawing board, they are of importance as bases for the study of new properties of the curve.

AN APPLICATION OF ABEL'S INTEGRAL EQUATION.

By W. C. BRENKE, University of Nebraska.

Let the shaded area in the figure represent the cross section of a weir notch, the cross section being symmetrical with respect to the x-axis. The quantity of flow through the notch per unit time will be given by



$$Q = C\int_0^h \sqrt{h-x}\, f(x)dx,$$

where the form of the notch is determined by $y = f(x)$; $x \geqq 0$.

Consider the problem of determining $f(x)$ so that the quantity of flow per unit of time shall be proportional to a given power of the depth of stream; *i.e.*, $Q = k'h^m$, $m > 0$. Hence we must find $f(x)$ from an integral equation of the form

$$\int_0^h \sqrt{h-x}\, f(x)dx = kh^m. \tag{1}$$

Differentiation with respect to h gives

$$\int_0^h \frac{f(x)}{\sqrt{h-x}}\,dx = 2kmh^{m-1}, \tag{2}$$

and a solution of (2) will be a solution of (1) also. But (2) comes under the form of Abel's integral equation,[1]

$$\int_a^x \frac{f(y)dy}{(x-y)^s} = g(x), \qquad (0 < s < 1),$$

which has the continuous solution

$$f(x) = \frac{\sin s\pi}{\pi}\int_a^x \frac{g'(y)dy}{(x-y)^{1-s}},$$

provided that $g(x)$ is continuous and has a finite derivative $g'(x)$ with at most a finite number of discontinuities in the range of integration, and that $g(a) = 0$. These conditions are satisfied in the problem under consideration if $m \geqq 2$, and hence we have

$$f\,(x) = \frac{2km(m-1)}{\pi}\int_0^x \frac{y^{m-2}dy}{\sqrt{x-y}}; \qquad m \geqq 2.$$

Evaluation of the last integral leads to a simple closed form for $f(x)$. By making the change of variable $y = xt$ we obtain

$$\int_0^x \frac{y^{m-2}dy}{\sqrt{x-y}} = x^{m-3/2}\int_0^1 \frac{t^{m-2}dt}{\sqrt{1-t}}. \tag{3}$$

But from the theory of the gamma-function[2] we have

$$\int_0^1 \frac{t^{p-1}dt}{(1-t)^{1-q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Applying this to (3) and substituting in the equation for $f(x)$, we get

$$f(x) = \frac{2km(m-1)}{\pi}\cdot\frac{\Gamma(m-1)\Gamma(\frac{1}{2})}{\Gamma(m-\frac{1}{2})}\,x^{m-\frac{3}{2}},$$

or, since $k\Gamma(k) = \Gamma(k+1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$f(x) = \frac{2k\Gamma(m+1)}{\sqrt{\pi}\,\Gamma(m-\frac{1}{2})}\,x^{m-\frac{3}{2}}; \qquad m \geqq 2. \tag{4}$$

When $m = n$, where n is a positive integer $\geqq 2$, we have

$$f(x) = \frac{k}{\pi}\cdot\frac{2^n\,n!}{1\cdot 3\cdot 5\cdot\,\cdots\,(2n-3)}\,x^{n-\frac{3}{2}}. \tag{5}$$

[1] M. Bôcher, *An Introduction to the Study of Integral Equations*, Cambridge, University Press, 1909, pp. 8–9.

[2] Nielsen, *Handbuch der Theorie der Gamma-Funktion*, 1906, p. 133.

When $m = n + 1/2$, n as above, we have

$$f(x) = k \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2n+1)}{2^n (n-1)!} x^{n-1}. \tag{6}$$

When $n = 2$, (5) gives the parabola, in which the quantity of flow is proportional to the square of the depth of stream, and (6) gives the triangular weir notch, with flow proportional to $h^{5/2}$.

It is easy now to show that (4) is still a solution of (1), though not necessarily continuous, for values of $m > 1/2$. To do this we put $x = ht$ in (1), which gives

$$\int_0^1 \sqrt{1-t}\, f(ht)dt = kh^{m-\frac{3}{2}}.$$

Substituting $f(ht) = ch^n t^n$ gives

$$c \int_0^1 \sqrt{1-t}\, t^n dt = kh^{m-n-\frac{3}{2}}.$$

The left member of this equation is independent of h, hence we must have

$$n = m - 3/2, \qquad \text{and} \qquad c \int_0^1 \sqrt{1-t}\, t^n dt = k,$$

which gives, since $\Gamma(3/2) = \sqrt{\pi}/2$,

$$c = \frac{2k}{\sqrt{\pi}} \cdot \frac{\Gamma(m+1)}{\Gamma(m-\frac{1}{2})}, \quad \text{provided } m > 1/2.$$

Hence (4) is a solution if $m > 1/2$.

When $m = 3/2$ we get the rectangular notch, $y =$ constant, and when $m = 1$ we get the curve $y = 1/\sqrt{x}$, such that the flow is directly proportional to the depth of stream.

DEPRECIATION BY A CONSTANT PERCENTAGE PLUS A CONSTANT.

By C. R. FORSYTH, Dartmouth College.

There are two ways in which a piece of property may depreciate which are usually considered in any complete treatise on the mathematical theory of depreciation, treatments in which possible interest accumulations are given no consideration. The methods employed to compute the annual or periodic allowance for depreciation corresponding to these two ways are known familiarly as the "straight line" method and the "constant percentage of book value" method. The annual allowance corresponding to the first case is the constant

$$k = \frac{C - S}{n}, \tag{1}$$

where C denotes the original cost, S the scrap value and n the estimated lifetime